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Integrable maps for the Garnier and for the Neumann system

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Abstract. We construct integrable maps for the Garnier and for the Neumann system. They are related to the Toda hierarchy exactly in the same way as the Garnier and the Neumann systems are related to the KdV hierarchy: as restricted flows. Here we give Lax representations for these maps and prove that they are completely integrable.

1. Introduction

Numerical evidence indicates [1] that integrable numerical schemes which are one-parameter families of integrable maps are superior with respect to ordinary discretizations or even symplectic discretizations of integrable Hamiltonian systems. It is particularly evident in the neighbourhood of a 'heteroclinic point' where subsequent magnifications of the neighbourhood do not display any signs of numerical chaos.

To find an integrable map for a given integrable mechanical system is a formidable task unless we have some additional structural information at our disposal. In this paper we show that a particularly useful piece of information is the knowledge that the mechanical system is a restricted flow of certain soliton hierarchy, as many integrable mechanical systems are known to be [2–4]. Then by a suitable discretization of the underlying spectral problem (see [5]) one obtains a lattice hierarchy of integrable equations. Restricted flows of this lattice hierarchy usually appear to be the desired discretizations which in the continuous limit go to the restricted flows of the continuous hierarchy.

In this paper we apply this procedure to find integrable maps for the first restricted flows of the KdV hierarchy: the Garnier and the Neumann systems. As a suitable discretization of the Schrödinger spectral problem we take the Toda spectral problem. By first studying restricted flows of the Toda hierarchy we find integrable maps for the Garnier and for the Neumann systems. In the following we shall call these maps the Garnier and the Neumann map. These maps appear to be rational, forward and explicit so that they are convenient for numerical implementation. They are also shown to go in the continuous limit to the Garnier and to the Neumann system. For these maps we derive a Lax representation and integrals of motion which are shown to be functionally independent and in involution. These maps are completely integrable since they are symplectic and live on N-dimensional invariant manifolds.

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2. Restricted flows of the KdV hierarchy

For the KdV hierarchy of equations we denote

$$\partial u/\partial t_m = K_m[u] = B_0 \delta H_m/\delta u = B_1 \delta H_{m-1}/\delta u \qquad m = 0, 1, 2, \ldots \tag{2.1}$$

where vector fields $K_m[u]$ are differential functions of u and its derivatives $u_x, u_{xx}, \ldots$. The first vector fields are: $K_0 = 0$, $K_1 = u_x$, $K_2 = \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x, \ldots$. The KdV hierarchy is bi-Hamiltonian: it can be expressed with the use of two Hamiltonian operators $B_0 = \partial = \partial/\partial x$, $B_1 = \frac{1}{4}\partial^3 + u\partial + \frac{1}{2}u_x$ acting on the Euler–Lagrange derivatives of the Hamiltonian densities $H_m[u]$. Equations (2.1) follow from the Schrödinger spectral problem

$$0 = (\partial^2 + u - \lambda)\phi \tag{2.2}$$

as the condition of compatibility with the auxiliary linear problem of the form $\partial\phi/\partial t_m = (\frac{1}{2}P_m[u,\lambda] + Q_m[u,\lambda])\phi$ where $P_m[u,\lambda]$, $Q_m[u,\lambda]$ are differential functions of u and polynomials in the spectral parameter λ. The spectral problem equation has two linearly independent solutions ϕ, ψ. They generate three linearly independent square eigenfunctions $\Psi_1 = \phi^2$, $\Psi_2 = \phi\psi$, $\Psi_3 = \psi^2$ satisfying the square eigenfunction relation (SER)

$$0 = \left[\tfrac{1}{4}\partial^3 + (u-\lambda)\partial + \tfrac{1}{2}u_x\right]\Psi_k = [B_1 - \lambda B_0]\Psi_k \qquad k = 1, 2, 3 \tag{2.3}$$

which is a differential consequence of (2.2).

Restricted flows of the KdV hierarchy are defined [2] as the set of $2N + 1$ equations

$$0 = (\partial^2 + u - \xi_k)\phi_k \qquad 0 = (\partial^2 + u - \xi_k)\psi_k \qquad k = 1, \ldots, N \tag{2.4a}$$

$$K_m[u] = \alpha B_0(\phi\psi) \qquad \text{(for fixed } m = 0, 1, 2, \ldots, \text{ and } \alpha = \text{constant)} \tag{2.4b}$$

where $(\phi\psi) = (\sum_{k=1}^N \phi_k\psi_k)$ and $\phi = (\phi_1, \ldots, \phi_N)^t$. They consists of $2N$ copies (2.4a) of the spectral problem taken for fixed values ξ_k of the spectral parameter λ and of the restriction (2.4b). Notice that if one takes, on the right-hand side of (4.4b), a more general linear combination $(\alpha(\phi\phi) + \beta(\phi\psi) + \gamma(\psi\psi))$ of square eigenfunctions, then it can be reduced to $\alpha(\phi\psi)$ by taking suitable linear combinations of ϕ and ψ. Usually, one considers the reduction $\phi = \psi$ which yields ODEs having a transparent mechanical interpretation. Here we hold ϕ and ψ distinct because the Garnier map (3.1) found below does not admit this symmetry reduction.

The first restricted flow ($m = 0$) in the vector notation is

$$0 = \phi_{xx} + (u - \Xi)\phi \qquad 0 = \psi_{xx} + (u - \Xi)\psi \qquad \Xi = \mathrm{diag}(\xi_1, \ldots, \xi_N) \tag{2.5a}$$

$$0 = (\phi\psi)_x \qquad \text{and hence} \quad (\phi\psi) = c = \text{constant}. \tag{2.5b}$$

In the reduction $\phi = \psi$ it is the Neumann system describing a particle moving on a sphere of radius $c > 0$ under the action of harmonic force.

The second restricted flow ($m = 1$)

$$0 = \phi_{xx} + (u - \Xi)\phi \qquad 0 = \psi_{xx} + (u - \Xi)\psi \tag{2.6a}$$

$$u_x = (\phi\psi)_x \qquad \text{and hence} \quad u = (\phi\psi) + \mathrm{c} \tag{2.6b}$$

yields the two-field Garnier system

$$0 = \phi_{xx} + (\phi\psi)\phi + (c - \Xi)\phi \qquad 0 = \psi_{xx} + (\phi\psi)\psi + (c - \Xi)\psi$$

originally considered by Garnier [6].

For these two restricted flows we find discrete integrable symplectic maps which approximate them in the continuous limit. These maps are derived by considering the Toda spectral problem which in the proper continuous limit goes to the Schrödinger spectral problem.

3. Garnier map

As the Garnier map we call the following set of $2N$ rational difference equations of second order:

$$0 = \phi_{n-1} + [(\phi_n \psi_n) - a - \Xi]\phi_n + \frac{b}{[(\phi_{n+1}\psi_n) - 1]}\phi_{n+1} = \frac{\partial L}{\partial \psi_n} + \boldsymbol{E}^{-1}\frac{\partial L}{\partial \psi_{n+1}} \tag{3.1a}$$

$$0 = \psi_{n+1} + [(\phi_n \psi_n) - a - \Xi]\psi_n + \frac{b}{[(\phi_n\psi_{n-1}) - 1]}\psi_{n-1} = \frac{\partial L}{\partial \phi_n} + \boldsymbol{E}^{-1}\frac{\partial L}{\partial \phi_{n+1}} \tag{3.1b}$$

for the vector variables ϕ_{n+1}, ψ_{n+1} where $\phi_n = (\phi_{n1}, \ldots, \phi_{nN})^t$, $\psi_n = (\psi_{n1}, \ldots, \psi_{nN})^t$ and a, b are arbitrary constants. Here the symbol $\boldsymbol{E}$ means the shift operator $\boldsymbol{E}\phi_n = \phi_{n+1}$. Equations (3.1) have the Lagrangian

$$L = (\phi_n \psi_{n+1}) + \tfrac{1}{2}(\phi_n \psi_n)^2 - (\phi_n(a + \Xi)\psi_n) + b \log[1 - (\phi_{n+1}\psi_n)] \tag{3.2}$$

where $(\phi_n(a + \Xi)\psi_n) = \sum_{k=1}^{N} \phi_{nk}(a + \Xi_k)\psi_{nk}$.

In order to justify the name Garnier map given to equations (3.1) we have to show that this map yields the Garnier system of equations in a certain continuous limit. For this purpose we rescale variables

$$\phi \to \Delta\phi \qquad \psi \to \Delta\psi \qquad a + \Xi \to 2 + \Delta^2\tilde{\Xi} \qquad \text{and set} \quad b = -1\,. \tag{3.3}$$

By expanding (with respect to small parameter Δ) functions $\phi_{n-1} = \Delta\phi(x - \Delta)$ and $\phi_{n+1} = \Delta\phi(x + \Delta)$ in (3.1a) we obtain

$$\begin{aligned}&\big[\Delta\phi - \Delta^2\phi_x + \tfrac{1}{2}\Delta^3\phi_{xx} + \mathrm{O}(\Delta^4)\big] + [\Delta^2(\phi\psi) - 2 - \Delta^2\tilde{\Xi}]\Delta\phi \\ &\qquad + b[\Delta^2(\phi\psi) + \Delta^3(\phi_x\psi) + \mathrm{O}(\Delta^4) - 1]^{-1}\big[\Delta\phi + \Delta^2\phi_x + \tfrac{1}{2}\Delta^3\phi_{xx} + \mathrm{O}(\Delta^4)\big] = 0\,.\end{aligned}$$

The Δ-order terms cancel and the Δ^2-order terms also cancel when $b = -1$ since $[\Delta^2(\phi\psi) + \Delta^3(\phi_x\psi) + \mathrm{O}(\Delta^4) - 1]^{-1} = -[1 + \Delta^2(\phi\psi) + \mathrm{O}(\Delta^3)]$. At Δ^3 we then obtain

$$\phi_{xx} + 2(\phi\psi)\phi - \tilde{\Xi} = 0\,. \tag{3.4a}$$

By a similar expansion of (3.1b) we get

$$\psi_{xx} + 2(\phi\psi)\psi - \tilde{\Xi}\psi = 0\,. \tag{3.4b}$$

Equations (3.4) are the two-field Garnier system [6]. In the same continuous limit for the Lagrangian (3.2) we recover at Δ^4 the Lagrangian $L = (\phi_x\psi_x) - (\phi\psi)^2 + (\phi\tilde{\Xi}\psi)$ modulo an exact derivative. In order to prove complete integrability of the map (3.1) we introduce canonical variables [7, 8] as

$$q_n = \phi_n \qquad p_n = \boldsymbol{E}^{-1}\frac{\partial L}{\partial \phi_{n+1}} = \frac{b}{[(\phi_n\psi_{n-1}) - 1]}\psi_{n-1} \tag{3.5}$$

$$r_n = \psi_n \qquad s_n = \boldsymbol{E}^{-1}\frac{\partial L}{\partial \psi_{n+1}} = \phi_{n-1} \tag{3.6}$$

where q_n, p_n, r_n, s_n are N-vectors defined in the same way as ϕ_n, ψ_n in the formulae (3.1). Equations (3.1) define a one step implicit map

$$0 = s_n + [(q_n r_n) - a - \Xi]q_n + \frac{b}{[(q_{n+1}r_n) - 1]}q_{n+1} \tag{3.7a}$$

$$0 = r_{n+1} + [(q_n r_n) - a - \Xi]r_n + p_n \tag{3.7b}$$

$$s_{n+1} = q_n \qquad p_{n+1} = \frac{b}{[(q_{n+1}r_n) - 1]}r_{n+1}\,. \tag{3.7c}$$

It can easily be solved for the variables with index $n+1$ by calculating $(q_{n+1}r_n)$ from (3.7*a*). The explicit form of this map is

$$r_{n+1} = -[(q_n r_n) - a - \Xi]r_n - p_n \qquad s_{n+1} = q_n \tag{3.8a}$$

$$p_{n+1} = [(s_n r_n) + (q_n r_n)^2 - a(q_n r_n) - \Xi(q_n r_n) + b][((q_n r_n) - a - \Xi)r_n + p_n] \tag{3.8b}$$

$$q_{n+1} = \frac{s_n + [(q_n r_n) - a - \Xi]q_n}{[(s_n r_n) + (q_n r_n)^2 - a(q_n r_n) - \Xi(q_n r_n) + b]}. \tag{3.8c}$$

This map is symplectic since $\{q_{\alpha n}, p_{\beta n}\} = \{r_{\alpha n}, s_{\beta n}\} = \{q_{\alpha(n+1)}, p_{\beta(n+1)}\} = \{r_{\alpha(n+1)}, s_{\beta(n+1)}\} = \delta_{\alpha\beta}$. Its discrete Lax representation is the similarity transformation

$$0 = \mathbb{A}_{n-1}(D\lambda + U_n) - (D\lambda + U_n)\mathbb{A}_n \tag{3.9}$$

with

$$(D\lambda + U_n) = \begin{pmatrix} \lambda - u_n & -v_n \\ 1 & 0 \end{pmatrix}$$

$$\mathbb{A}_{n-1} = \begin{pmatrix} -\sum_{\alpha=1}^N \frac{q_{n\alpha}p_{n\alpha}}{\lambda-\xi_\alpha} & v_{n-1} + \sum_{\alpha=1}^N \frac{p_{n\alpha}s_{n\alpha}}{\lambda-\xi_\alpha} \\ -1 - \sum_{\alpha=1}^N \frac{q_{n\alpha}r_{n\alpha}}{\lambda-\xi_\alpha} & \sum_{\alpha=1}^N \frac{r_{n\alpha}s_{n\alpha}}{\lambda-\xi_\alpha} + (\lambda - u_n) + (q_n r_n) \end{pmatrix} \tag{3.10}$$

where $u_n = (q_n r_n) - a$ and $v_{n-1} = (q_n p_n) - b$. From $\mathrm{Tr}\,\mathbb{A}_{n-1}$ we get N integrals of motion

$$J_\alpha = r_\alpha s_\alpha - q_\alpha p_\alpha \qquad \alpha = 1, \ldots, N \tag{3.11a}$$

at the poles ξ_α. They reflect the permutational symmetry with respect to the interchange of ϕ and ψ. Det $\mathbb{A}_{n-1}$ gives N further integrals

$$\begin{aligned} K_\alpha = &- (\xi_\alpha + a)q_\alpha p_\alpha + p_\alpha s_\alpha + [(qp) - b]q_\alpha r_\alpha + \sum_{\gamma=1,\gamma\neq\alpha}^N \frac{1}{\xi_\alpha - \xi_\gamma}(p_\alpha r_\gamma - p_\gamma r_\alpha) \\ &\times (s_\alpha q_\gamma - s_\gamma q_\alpha) \\ \equiv\ & P_\alpha + L_\alpha \end{aligned} \tag{3.11b}$$

at the poles ξ_α. We denote $P_\alpha = -q_\alpha(\xi_\alpha + a)p_\alpha + p_\alpha s_\alpha + [(qp) - b]q_\alpha r_\alpha$. The index n is suppressed in all canonical variables. Integrals (J_α, K_α) are functionally independent and in involution whenever $\xi_\alpha \neq \xi_\beta$ for $\alpha \neq \beta$. For proving functional independence of integrals (3.11) we consider the Jacobian $\partial(J, K)/\partial(r, s)$ with the parameters rescaled as $\xi_\alpha \to \rho\xi_\alpha$. When $\rho \to \infty$ then the leading term is

$$\mathrm{Det}\frac{\partial(J, K)}{\partial(r, s)} = \left[\prod_{\alpha=1}^N s_\alpha p_\alpha - \prod_{\beta=1}^N r_\beta q_\beta\right] + \mathrm{O}(\rho^{-1}) \neq 0$$

except, perhaps, some singular lower-dimensional manifolds in the phase space.

It is true for all values of ξ_α since this is an algebraic property of the Jacobian. The proof of involutivity of integrals (J_α, K_α) is more complicated. One first shows that

$$\{L_\alpha, L_\beta\} = 0, \{(qp), L_\alpha\} = 0$$

and then finds

$$\{K_\alpha, K_\beta\} = \{P_\alpha, P_\beta\} + \{P_\alpha, L_\beta\} + \{L_\alpha, P_\beta\} = 0\,.$$

4. Neumann map

Here we call the Neumann map the following set of $2N$ rational difference equations of second order:

$$\phi_{n-1} + (u_n - \Xi)\phi_n + d(\phi_{n+1}\psi_n)^{-1}\phi_{n+1} = 0 \tag{4.1a}$$

$$\psi_{n+1} + (u_n - \Xi)\psi_n + d(\phi_n\psi_{n-1})^{-1}\psi_{n-1} = 0 \tag{4.1b}$$

where $(\phi_n\psi_n) = c =$ constant and the Lagrangian multiplier $cu_n = (\phi_n \Xi \psi_n) - (\phi_{n-1}\psi_n) - d$ is determined from this constraint ($d =$ constant). These equations have the discrete Lax representation (3.10) with the same $(D\lambda + U_n)$ as for the Garnier map but with the Lax matrix

$$\mathbb{A}_{n-1} = \begin{pmatrix} -v_{n-1}\sum_{\alpha=1}^{N}\frac{\phi_{n\alpha}\psi_{(n-1)\alpha}}{\lambda-\xi_\alpha} & v_{n-1}\sum_{\alpha=1}^{N}\frac{\phi_{(n-1)\alpha}\psi_{(n-1)\alpha}}{\lambda-\xi_\alpha} \\ -\sum_{\alpha=1}^{N}\frac{\phi_{n\alpha}\psi_{n\alpha}}{\lambda-\xi_\alpha} & (\phi_n\psi_n) + \sum_{\alpha=1}^{N}\frac{\phi_{(n-1)\alpha}\psi_{n\alpha}}{\lambda-\xi_\alpha} \end{pmatrix} \tag{4.2}$$

where $v_{n-1} = d(\phi_n\psi_{n-1})^{-1}$. It yields the following integrals of motion for (4.1):

$$J_\alpha = \phi_{(n-1)\alpha}\psi_{n\alpha} - v_{n-1}\phi_{n\alpha}\psi_{(n-1)\alpha} \tag{4.3a}$$

$$K_\alpha = v_{n-1}\Bigg[\sum_{\gamma=1,\gamma\neq\alpha}^{N}\frac{1}{\xi_\alpha-\xi_\gamma}(\psi_{(n-1)\alpha}\psi_{n\gamma} - \psi_{(n-1)\gamma}\psi_{n\alpha})(\phi_{(n-1)\alpha}\phi_{n\gamma} - \phi_{(n-1)\gamma}\phi_{n\alpha}) + c\phi_{n\alpha}\psi_{(n-1)\alpha}\Bigg] \qquad \alpha = 1,\ldots,N\,. \tag{4.3b}$$

They are not functionally independent: $\sum_{\alpha=1}^{N} K_\alpha = cd$ since we are dealing with a constrained system.

In order to justify the name Neumann map we recover the Neumann system in the continuous limit. Let us rescale $\Xi \to 2 + \Delta^2\tilde{\Xi}$ and take $d = c$. Then by expanding functions $\phi_{n-1} = \Delta\phi(x-\Delta)$ and $\phi_{n+1} = \Delta\phi(x+\Delta)$ in the equation (4.1*a*) and by using $(\phi_n\psi_n) = c$ we obtain the following system of ODEs:

$$\phi_{xx} - \tilde{\Xi}\phi + \nu_1\phi - \frac{1}{c}(\phi_x\psi)\phi_x = 0 \tag{4.4a}$$

$$\psi_{xx} - \tilde{\Xi}\psi + \nu_2\psi - \frac{1}{c}(\psi_x\phi)\psi_x = 0 \tag{4.4b}$$

where $\nu_1 = \frac{1}{c}[(\phi\tilde{\Xi}\psi) - (\psi\phi_{xx}) + \frac{1}{c}(\phi_x\psi)^2]$ and $\nu_2 = \frac{1}{c}[(\phi\tilde{\Xi}\psi) - (\phi\psi_{xx}) + \frac{1}{c}(\psi_x\phi)^2]$ are determined from the constraint $(\phi_n\psi_n) = c$. Equations (4.4*a, b*) are the two-field Neumann system. They admit the reduction $\phi = \psi$ which in turn, due to the constraint $(\phi\psi) = c$, implies $(\phi_x\psi) = (\psi_x\phi) = 0$. In this reduction equations (4.4) become the classical Neumann system

$$\phi_{xx} - \tilde{\Xi}\phi + \nu\phi = 0 \tag{4.5}$$

with $\nu = \frac{1}{c}[(\phi\tilde{\Xi}\phi) - (\phi\phi_{xx})] = \frac{1}{c}[(\phi\tilde{\Xi}\phi) - (\phi_x\phi_x)]$. Also, integrals of motion admit a continuous limit and in the reduction $\phi = \psi$ we recover the Uhlenbeck integrals

$$K_\alpha = -c\phi_\alpha^2 + \sum_{\gamma=1,\gamma\neq\alpha}^{N}\frac{1}{\xi_\alpha-\xi_\gamma}(\phi_\alpha\phi_{\gamma x} - \phi_\gamma\phi_{\alpha x})^2\,.$$

For proving complete integrability of the map (4.1) we have to introduce suitable canonical variables.

The Neumann map (4.1) is the stationarity condition for the discrete action

$$S = \sum_{n\in\mathbf{Z}}(\phi_n\psi_{n+1}) - (\phi_n\Xi\psi_n) + d\log(\phi_{n+1}\psi_n)$$

on the manifold $(\phi_n\psi_n) = c$. We shall choose the Lagrangian $L_n = (\phi_n\psi_{n+1}) - (\phi_n\Xi\psi_n) + d\,\log(\phi_{n+1}\psi_n)$ to define 'unconstrained' canonical variables as

$$q_n = \phi_n \qquad p_n = \boldsymbol{E}^{-1}\frac{\partial L_n}{\partial\phi_{n+1}} = \frac{d\psi_{n-1}}{(\phi_n\psi_{n-1})}\psi_{n-1} = v_{n-1}\psi_{n-1} \tag{4.6a}$$

$$r_n = \psi_n \qquad s_n = \boldsymbol{E}^{-1}\frac{\partial L_n}{\partial\psi_{n+1}} = \phi_{n-1}\,. \tag{4.6b}$$

The constraints then are $(q_n r_n) = c$ and $(p_n q_n) = d$. On the manifold of constraints the inverse of the map (4.6) is $\phi_n = q_n$, $\psi_n = r_n$, $\phi_{n-1} = s_n$, $\psi_{n-1} = cp_n/(p_n s_n)$ and in terms of (q, p, r, s) the Neumann map reads

$$r_{n+1} + (u_n - \Xi)r_n + p_n = 0 \qquad s_{n+1} = q_n \tag{4.7a}$$

$$p_{n+1} = (d/\alpha_n)r_n \qquad s_n + (u_n - \Xi)q_n + (d/\alpha_n)q_{n+1} = 0 \qquad \alpha_n = q_{n+1}r_n\,. \tag{4.7b}$$

The constraint $(q_{n+1}r_{n+1}) = c$ determines $(dc/\alpha_n) = ([s_n + (u_n - \Xi)q_n], [p_n + (u_n - \Xi)r_n])$ while $(p_{n+1}q_{n+1}) = d$ determines $cu_n = (r_n\Xi q_n) - (r_n s_n) - d$. The symbol $([\cdot],[\cdot])$ reads here as a scalar product. So the explicit and forward form of the Neumann map is

$$q_{n+1} = -\frac{c[s_n + (u_n - \Xi)q_n]}{([s_n + (u_n - \Xi)q_n], [p_n + (u_n - \Xi)r_n])} \qquad r_{n+1} = -[p_n + (u_n - \Xi)r_n] \tag{4.8a}$$

$$p_{n+1} = c^{-1}([s_n + (u_n - \Xi)q_n], [p_n + (u_n - \Xi)r_n])r_n \qquad s_{n+1} = q_n\,. \tag{4.8b}$$

It can be checked that (4.8) is a Poisson map with respect to the Dirac bracket on the manifold of the constraints: $(p_n q_n) = d$, $(q_n r_n) = c$. Now the complete integrability follows from the existence of integrals (4.3) rewritten in terms of canonical variables as

$$J_j = r_j s_j - q_j p_j \qquad j = 1, \ldots, N \tag{4.9a}$$

$$K_j = \sum_{k=1,k\neq j}^{N} \frac{1}{\xi_j - \xi_k}(p_j r_k - p_k r_j)(s_j q_k - s_k q_j) - cq_j p_j\,. \tag{4.9b}$$

Together with the function $(qp) = \sum_{j=1}^{N} q_j p_j$ they form a complete set of $2N$ functionally independent integrals in involution and the map (4.8) is completely integrable in the 'unconstrained' space of variables (q, p, r, s) [9]. Clearly it remains integrable on the manifold of constraints $(q_n r_n) = c$, $(p_n q_n) = d$ which is preserved by (4.8). Incidentally, a similar set of involutive functions appeared in connection with the second restricted flow of the AKNS hierarchy [10].

5. The Toda hierarchy

The Garnier and Neumann maps has been derived as restricted flows of the Toda hierarchy. But it is more convenient to think of them as stationary flows of the Toda hierarchy with sources because as such they inherit their Lax representation.

Let us fix notation and consider the Toda spectral problem

$$(\mathbb{L} - \lambda)\phi = [\boldsymbol{E}^{-1} + (u - \lambda)\boldsymbol{I} + v\boldsymbol{E}]\phi = 0 \tag{5.1a}$$

where $u = u_n$, $v = v_n$ are the Toda fields, $\phi = \phi_n$ are eigenfunctions, λ is the spectral parameter (independent of n), $\boldsymbol{I}$ = identity and $\boldsymbol{E}$ is the shift operator $\boldsymbol{E}\phi_n = \phi_{n+1}$ which acts on all functions to the right which depend on the index n. Index n is in the sequel reserved for the lattice components and we will omit it unless it is shifted.

The adjoint Toda spectral problem reads

$$[\boldsymbol{E} + (u-\lambda)\boldsymbol{I} + \boldsymbol{E}^{-1}v]\psi = 0 \tag{5.1b}$$

where the adjoint is taken with respect to the pairing $\langle \phi, \psi \rangle = \sum_{n=-\infty}^{\infty} \phi_n \psi_n$.

The Toda hierarchy arises as the condition of compatibility with the auxiliary linear problem written as

$$\phi_t = v(P\boldsymbol{E} - Q)\phi\,. \tag{5.2}$$

A sufficient condition for the existence of common solutions ϕ to (5.1*a*) and (5.2) is

$$\begin{pmatrix} u \\ v^t \end{pmatrix} = \begin{pmatrix} v\boldsymbol{E} - \boldsymbol{E}^{-1}v & (u-\lambda)(\boldsymbol{I} - \boldsymbol{E}^{-1})v \\ v(\boldsymbol{E} - \boldsymbol{I})(u-\lambda) & v(\boldsymbol{E} - \boldsymbol{E}^{-1})v \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix} = (B_1 - \lambda B_0) \begin{pmatrix} P \\ Q \end{pmatrix} \tag{5.3}$$

where B_0 and B_1 denote 2×2 Hamiltonian operators which we can identify at λ and $\lambda^0 = 1$. The usual Toda hierarchy follows from the polynomial assumption for $P(\lambda) = P_{(m)} = \sum_{k=0}^{m} \lambda^k P_{m-k}$ and $Q(\lambda) = Q_{(m)} = \sum_{k=0}^{m} \lambda^k Q_{m-k}$. The requirement of λ independence of vector fields (5.3) determines all P_r and Q_r, $r = 0, 1, 2, \ldots$ from the recursion

$$0 = (v\boldsymbol{E} - \boldsymbol{E}^{-1}v)P_m + u(\boldsymbol{I} - \boldsymbol{E}^{-1})vQ_m - (\boldsymbol{I} - \boldsymbol{E}^{-1})vQ_{m+1} \tag{5.4a}$$

$$0 = v(\boldsymbol{E} - \boldsymbol{I})uP_m + v(\boldsymbol{E} - \boldsymbol{E}^{-1})vQ_m - v(\boldsymbol{E} - \boldsymbol{I})P_{m+1}\,. \tag{5.4b}$$

The first of these are

$$\begin{aligned} &P_0 = 1 \qquad P_1 = u_n \qquad P_2 = u_n^2 + v_n + v_{n-1} \\ &P_3 = u_n(u_n^2 + v_n + v_{n-1}) + v_n(u_n + u_{n+1}) + v_{n-1}(u_n + u_{n-1}), \ldots \end{aligned} \tag{5.5a}$$

and

$$\begin{aligned} &Q_0 = 0 \qquad Q_1 = 1 \qquad Q_2 = u_n + u_{n+1} \\ &Q_3 = u_{n+1}^2 + u_n u_{n+1} + u_n^2 + v_{n+1} + v_n + v_{n-1}, \ldots. \end{aligned} \tag{5.5b}$$

The corresponding vector fields read

$$\begin{aligned} &K_0 = B_0 \begin{pmatrix} P_0 \\ Q_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad K_1 = \begin{pmatrix} v_n - v_{n-1} \\ v_n(u_{n+1} - u_n) \end{pmatrix} \\ &K_2 = \begin{pmatrix} v_n(u_n + u_{n+1}) - v_{n-1}(u_n + u_{n-1}) \\ v_n(u_{n+1}^2 - u_n^2 + v_{n+1} - v_{n-1}) \end{pmatrix} \ldots. \end{aligned} \tag{5.6}$$

By the square eigenfunction relation (SER) we mean the equation

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} v\boldsymbol{E} - \boldsymbol{E}^{-1}v & (u-\lambda)(\boldsymbol{I} - \boldsymbol{E}^{-1})v \\ v(\boldsymbol{E} - \boldsymbol{I})(u-\lambda) & v(\boldsymbol{E} - \boldsymbol{E}^{-1})v \end{pmatrix} \begin{pmatrix} \phi\psi \\ (\boldsymbol{E}\phi)\psi \end{pmatrix} = (B_1 - \lambda B_0) \begin{pmatrix} \phi\psi \\ (\boldsymbol{E}\phi)\psi \end{pmatrix} \tag{5.7}$$

which is a consequence of the spectral problem (5.1*a*) and of its adjoint (5.1*b*).

6. Toda hierarchy with sources

The Toda hierarchy with sources follows from the rational ansatz for

$$P(\lambda) = P_{(m)} + P_{rat} = \sum_{k=0}^{m} \lambda^k P_{m-k} + \sum_{r=1}^{N} (\lambda - \xi_r)^{-1} p^{(r)} \tag{6.1a}$$

$$Q(\lambda) = Q_{(m)} + Q_{rat} = \sum_{k=0}^{m} \lambda^k Q_{m-k} + \sum_{r=1}^{N} (\lambda - \xi_r)^{-1} q^{(r)}\,. \tag{6.1b}$$

Then, due to the identity

$$[B_1 - \lambda B_0](\lambda - \xi)^{-1}q = [B_1 - (\lambda - \xi)B_0 - \xi B_0](\lambda - \xi)^{-1}q$$
$$= -B_0 q + [B_1 - \xi B_0](\lambda - \xi)^{-1}q$$

equation (5.3) splits into the equation

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = (B_1 - \lambda B_0)\begin{pmatrix} P_{(m)} \\ Q_{(m)} \end{pmatrix} - B_0\begin{pmatrix} \sum_{r=1}^N p^{(r)} \\ \sum_{r=1}^N q^{(r)} \end{pmatrix} \tag{6.2}$$

which we obtain as the entire part and into the set of square eigenfunction relations

$$(B_1 - \xi_r B_0)\begin{pmatrix} p^{(r)} \\ q^{(r)} \end{pmatrix} = 0 \qquad r = 1, \ldots, N \tag{6.3}$$

at $(\lambda - \xi_r)^{-1}$. The right-hand side of equation (6.2) consists of the standard recursion relation at powers $\lambda, \ldots, \lambda^{m+1}$ which determines all P_{m-k}, Q_{m-k}, $k = 0, \ldots, m$ and of the dynamical vector field for the mth Toda flow which is supplemented with the source term depending on $p^{(r)}$, $q^{(r)}$.

The remaining n equations can be interpreted as constraints which acquire the meaning of the square eigenfunction relations if we set $p_n^{(r)} = \phi_n^{(r)}\psi_n^{(r)}$ and $q_n^{(r)} = \phi_{n+1}^{(r)}\psi_n^{(r)}$. A matrix form of Toda hierarchy arises when we introduce the column vector notation $\Phi = \Phi_n = (\phi_n, \phi_{n+1})^t$. We can rewrite then the linear problem equations (5.1*a*) and (5.2) as

$$\boldsymbol{E}^{-1}\Phi = \begin{pmatrix} \lambda - u & -v \\ 1 & 0 \end{pmatrix}\Phi = (D\lambda + U)\Phi \tag{6.4a}$$

$$\Phi_t = \begin{pmatrix} -vQ & vP \\ -(\boldsymbol{E}P) & -\boldsymbol{E}[vQ + (u - \lambda)P] \end{pmatrix}\Phi = \mathbb{A}\Phi \tag{6.4b}$$

and the compatibility condition reads

$$U_t = (D\lambda + U)_t = (\boldsymbol{E}^{-1}\mathbb{A})(D\lambda + U) - (D\lambda + U)\mathbb{A}\,. \tag{6.5}$$

7. Restricted flows of Toda hierarchy

A restricted flow of the Toda hierarchy (5.3) ($P = P_{(m)}$, $Q = Q_{(m)}$) is a set of $2N + 2$ difference equations. It consists of N copies (7.1*a*) of the spectral problem equations, taken for fixed values ξ_j of the spectral parameter λ, of N copies of its adjoint (7.1*b*) and of the restriction (7.1*c*) of the mth vector field of the Toda hierarchy:

$$[\boldsymbol{E}^{-1} + (u - \xi_\alpha)\boldsymbol{I} + v\boldsymbol{E}]\phi_\alpha = 0 \tag{7.1a}$$

$$[\boldsymbol{E} + (u - \xi_\alpha)\boldsymbol{I} + \boldsymbol{E}^{-1}v]\psi_\alpha = 0 \qquad \alpha = 1, \ldots, N \tag{7.1b}$$

$$0 = B_1\begin{pmatrix} P_m \\ Q_m \end{pmatrix} - B_0\begin{pmatrix} \sum_{\alpha=1}^N \phi_\alpha\psi_\alpha \\ \sum_{\alpha=1}^N (\boldsymbol{E}\phi_\alpha)\psi_\alpha \end{pmatrix} = B_0\begin{pmatrix} P_{m+1} \\ Q_{m+1} \end{pmatrix} - B_0\begin{pmatrix} (\phi\psi) \\ ((\boldsymbol{E}\phi)\psi) \end{pmatrix}. \tag{7.1c}$$

Here we use the shorthand scalar product notation $(\phi\psi) = \sum_{\alpha=1}^N \phi_\alpha\psi_\alpha$, $((\boldsymbol{E}\phi)\psi) = \sum_{\alpha=1}^N (\boldsymbol{E}\phi_\alpha)\psi_\alpha$. Equations (7.1) define a map for the variables ϕ_α, ψ_α, $\alpha = 1, \ldots, N$, which is expected to be integrable since equations (7.1) are invariant with respect to the action of all flows of the Toda hierarchy. We recall that the functions ϕ_α, ψ_α have a dumb index n which is usually suppressed in our notation. Equations (7.1*a*, *b*) are invariant by the definition of the hierarchy while (7.1*c*) can be seen as a stationarity condition

$$0 = B_0\begin{pmatrix} \delta H_{m+1}/\delta u - \sum_{r=1}^N \delta\xi_r/\delta u \\ \delta H_{m+1}/\delta v - \sum_{r=1}^N \delta\xi_r/\delta v \end{pmatrix} \tag{7.2}$$

of a flow (also belonging to the hierarchy) which involves the square eigenfunctions and the Hamiltonian H_{m+1}. The mth restricted flow of the Toda hierarchy can now be seen as the mth stationary flow of the Toda hierarchy with sources. It inherits from (6.5) the Lax representation

$$0 = (\boldsymbol{E}^{-1}\mathbb{A}_{(m)})(D\lambda + U) - (D\lambda + U)\mathbb{A}_{(m)} \tag{7.3}$$

which becomes a similarity transformation of $\mathbb{A}_{(m)}$. Restricted flows are numbered by $m = -1, 0, 1, 2, \ldots$.

The $m = 0$ restriction for the Toda hierarchy reads

$$0 = B_0 \begin{pmatrix} P_1 - (\phi\psi) \\ P_0 - ((\boldsymbol{E}\phi)\psi) \end{pmatrix} = \begin{pmatrix} 0 & (\boldsymbol{I} - \boldsymbol{E}^{-1})v \\ v(\boldsymbol{E} - \boldsymbol{I}) & 0 \end{pmatrix} \begin{pmatrix} u + a - (\phi\psi) \\ 1 + bv^{-1} - ((\boldsymbol{E}\phi)\psi) \end{pmatrix} \tag{7.4}$$

since $(u + a, 1 + bv^{-1})^t$ belongs to the kernel of B_0. It yields $u_n = (\phi_n\psi_n) - a$ and $v_n = b[(\phi_{n+1}\psi_n) - 1]^{-1}$ where a, b are constants of integration. The replicas of the spectral problem (5.1) become the Garnier map (3.1).

By specifying Q, P in (6.4) as $Q = \sum_{\alpha=1}^N \frac{q^{(\alpha)}}{\lambda - \xi_\alpha}$, $P = 1 + \sum_{\alpha=1}^N \frac{p^{(\alpha)}}{\lambda - \xi_\alpha}$ with $p_n^{(\alpha)} = \phi_{n\alpha}\psi_{n\alpha}$, $q_n^{(\alpha)} = \phi_{(n+1)\alpha}\psi_{n\alpha}$ we get the following Lax matrices for the Garnier map:

$$(D\lambda + U_n) = \begin{pmatrix} \lambda - u_n & -v_n \\ 1 & 0 \end{pmatrix}$$

and

$$\begin{aligned} \mathbb{A}_{n-1} &= \begin{pmatrix} -v_{n-1}Q_{n-1} & v_{n-1}P_{n-1} \\ -P_n & -[v_nQ_n + (u_n - \lambda)P_n] \end{pmatrix} \\ &= \begin{pmatrix} -v_{n-1}\sum_{\alpha=1}^N \frac{\phi_{n\alpha}\psi_{(n-1)\alpha}}{\lambda - \xi_\alpha} & v_{n-1}\left(1 + \sum_{\alpha=1}^N \frac{\phi_{(n-1)\alpha}\psi_{(n-1)\alpha}}{\lambda - \xi_\alpha}\right) \\ -1 - \sum_{\alpha=1}^N \frac{\phi_{n\alpha}\psi_{n\alpha}}{\lambda - \xi_\alpha} & -\left[v_{n-1}\sum_{\alpha=1}^N \frac{\phi_{(n+1)\alpha}\psi_{n\alpha}}{\lambda - \xi_\alpha} + (u_n - \lambda)\left(1 + \sum_{\alpha=1}^N \frac{\phi_{n\alpha}\psi_{n\alpha}}{\lambda - \xi_\alpha}\right)\right] \end{pmatrix}. \end{aligned}$$

By using the spectral problem (3.1*a*), in the matrix element (22), the spectral problem (7.1*a*) and by substituting definitions (3.5), (3.6) we get the Lax matrix (3.10). It is not difficult to check that the discrete Lax equation (3.9) is now satisfied due to equations (3.7) when u_n and v_n are substituted from the restrictions $u_n = (\phi_n\psi_n) - a = (q_nr_n) - a$ and $v_{n-1} = b[(\phi_{n+1}\psi_n) - 1]^{-1}(q_np_n) - b$.

The $m = -1$ restriction

$$0 = B_0 \begin{pmatrix} (\phi\psi) \\ ((\boldsymbol{E}\phi)\psi) \end{pmatrix} = \begin{pmatrix} 0 & (\boldsymbol{I} - \boldsymbol{E}^{-1})v \\ v(\boldsymbol{E} - \boldsymbol{I}) & 0 \end{pmatrix} \begin{pmatrix} (\phi\psi) \\ ((\boldsymbol{E}\phi)\psi) \end{pmatrix} \tag{7.5}$$

yields $(\phi_n\psi_n) = c$, $v_n = d(\phi_{n+1}\psi_n)^{-1}$ where c, d are constants of integration and the restricted flow equations become the discrete Neumann map (4.1). By specifying Q, P in (6.4) as $Q = \sum_{\alpha=1}^N \frac{q^{(\alpha)}}{\lambda - \xi_\alpha}$, $P = \sum_{\alpha=1}^N \frac{p^{(\alpha)}}{\lambda - \xi_\alpha}$ with $p_n^{(\alpha)} = \phi_{n\alpha}\psi_{n\alpha}$, $q_n^{(\alpha)} = \phi_{(n+1)\alpha}\psi_{n\alpha}$ we get the Lax matrix (4.2) for the Neumann map (4.1).

8. Conclusions

We have given here only the first two maps following from the Toda hierarchy. One expects that the next restricted flows of the Toda hierarchy ($m = 1, 2, \ldots$) should approximate the corresponding restricted flows ($m = 2, 3, \ldots$) of the KdV hierarchy. It would be particularly interesting to derive an integrable map approximating the integrable case of the Henon–Heiles system which is embedded into the $m = 3$ stationary KdV flow [3, 11].

We have to mention here that different integrable maps are presently available in the literature which are called Neumann and Garnier maps. Two of these have been derived by one of the authors [5, 13]. They have been derived as restricted flows of an alternative version of the Toda hierarchy associated with the self-adjoint spectral problem. An integrable discretization of the Neumann system has been presented by Moser and Veselov [12], while recently Suris [14] has presented yet another integrable discretization of the Garnier system. The models studied in [13, 14] are endowed with Lax representation and R-matrix structure.

Further properties of our models, like R-matrix formulation, bi-Hamiltonian structure and separation of variables are under study and the results will be published elsewhere.

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